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Dilatonic Randall-Sundrum theory and renormalization group

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Abstract: We extend Randall-Sundrum dynamics to non-conformal metrics corresponding to non-constant dilaton. We study the appareance of space-time naked singularities and the renormalization group evolution of four-dimensional Newton constant.

Keywords: p-branes, String Duality, Renormalization Regularization and Renormalons.

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Contents

1. Introduction

The idea that newtonian gravity can be localized in a three-brane world has received a lot of attention during the last year [1]–[8]. The original framework is a five-dimensional warped spacetime metric of the type:

$$ds^2 = A^2(z) d\vec{x}^2 - dz^2 \,, \tag{1.1}$$

with the warping factor A depending exponentially on z, more precisely a AdS_5 space-time with negative cosmological constant Λ. Small gravitational fluctuations $h_{\mu\nu}$ of the metric can be written as superpositions of modes $h_{\mu\nu} = e^{ipx}\psi(z)\epsilon_{\mu\nu}$. Four-dimensional gravitons are associated with zero modes defined by the condition $p^2 = 0$. In order to get normalizable zero modes we need to cutoff the deep ultraviolet region of AdS_5. This can be done introducing a domain wall at some finite value z. On the domain wall metric we can now generically have gravitational zero modes that can be interpreted as bound states of the higher dimensional graviton strongly localized around the wall. This is in summary the dynamical mechanism suggested in [1, 2] to induce four-dimensional newtonian gravity in a brane world. If we start with AdS_5 space-time the resulting domain wall metric will have a horizon at infinity. Very likely this horizon does not have any observable effect on the physics on the brane due to the strong redshift.

From the point of view of holography [9]–[11], the Randall-Sundrum mechanism of inducing gravity by introducing an ultraviolet cutoff could be interpreted as the extension of the holographic map to conformal field theories coupled to gravity [12, 13].

In this letter we will address the question of extending the RS- scenario to dilatonic gravity. One reason for that is of course to make a more direct contact with string theory where the dilaton appears naturally in the definition of brane tensions. Another reason is to unravel how much of RS-dynamics depends on conformal invariance. Once we include the dilaton we have at our disposal the possibility of working

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with a vanishing five-dimensional cosmological constant. In this case we find bound states four-dimensional gravitons with the Newton constant fine tuned in terms of the wall tension. For non-vanishing cosmological constant we find a two-parameter family of solutions depending on the dilaton coupling and on the cosmological constant. The physics of all these cases is different from that in RS- model in the sense that in the bulk there appears a naked singularity that can be reached from the wall in finite time. This singularity can only be avoided in the conformal AdS case. This occurs as an effect of working with a non-constant dilaton.

2. Construction of the solutions

Our starting point is the following five-dimensional action of a dilaton ϕ coupled to gravity in the presence of a cosmological constant Λ:

$$S_{\text{grav}} = \frac{1}{\kappa} \int d^4x dz \sqrt{|G|} \left[R - \frac{4}{3}(\partial\phi)^2 - e^{a\phi/3}\Lambda \right], \tag{2.1}$$

where a is a free parameter that determines the coupling of the dilaton to the cosmological constant. The domain wall solutions of this action are given by:

$$\begin{aligned} ds^2 &= [N(a)(z+z_0)]^{32/a^2} \, d\vec{x}^2 - dz^2 \,, \\ e^{-2\phi} &= [N(a)(z+z_0)]^{12/a} \,, \end{aligned} \tag{2.2}$$

where z runs between 0 and infinity, z_0 is integration constant, $N(a)$ a function of the coupling a and the cosmological constant:

$$N(a) = \frac{a^2}{12}\sqrt{\frac{3\Lambda}{(a^2-64)}} \,. \tag{2.3}$$

Since we assume the cosmological constant to be negative, our solution only makes sense for a's with values between 8 and -8. The above metric has clearly a naked singularity at $z = -z_0$:

$$\begin{aligned} R &= 2^7 \frac{40 - a^2}{a^4 (z+z_0)^2} \\ R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} &= 2^{13} \frac{640 - 32a^2 + a^4}{a^8 (z+z_0)^4} \,. \end{aligned} \tag{2.4}$$

To calculate the profile of the graviton we add some small fluctuations $h_{\mu\nu}$ to the above background metric, choosing the gauge $h^\mu_\mu = \partial^\mu h_{\mu\nu} = h_{z\nu} = h_{zz} = 0$:

$$ds^2 = [N(a)(z+z_0)]^{32/a^2} (\eta_{mn} + h_{mn}) dx^m dx^n - dz^2 \,, \tag{2.5}$$

Inserting the perturbed metric (2.5) in the equations of motion of (2.1), we get, at first order in the perturbation, the following differential equation for the graviton:

$$\left[\frac{1}{2}\partial^2 + \frac{16}{a^2}\cdot\frac{32-a^2}{a^2}(z+z_0)^{-2}\right] h_{mn} = 0 \tag{2.6}$$

For the splitting of variables $h_{\mu\nu}(x,z) = e^{i\vec{k}\vec{x}}\psi(z)\epsilon_{\mu\nu}$, we find for the profile of the graviton zero mode

$$\psi(z) = [N(a)(z+z_0)]^{32/a^2} \ . \tag{2.7}$$

This zero mode is not normalizable over the whole of space. If we insist on the existence of a graviton zero mode, we need to introduce a cut-off at the position $z = N(a)^{-1}$.

The cutt-off has the effect of throwing away the part of space with $z > N(a)^{-1}$, where the graviton zero mode (2.7) becomes non-normalizable. We can replace this thrown away part by a copy of the part of space with $z < N(a)^{-1}$. At the level of the solution (2.2), this is seen in the fact that we pass from the variables $z \to N(a)^{-1} - |z|$, (where $|z|$ now runs between 0 and $N(a)^{-1} - z_0$), where $N(a)z_0 \in [0,1]$. This generates a delta function behaviour in the equations of motion, which can be compensated by introducing domain wall source terms at the bounderies:

$$S_{\text{source}} = \int_{z=0} d^4x\sqrt{|\tilde{g}|}\left[\mathcal{L}_{\text{brane}} + e^{b\phi/3}V_0\right] + \int_{z=N(a)^{-1}} d^4x\sqrt{|\tilde{g}|}\left[\mathcal{L}_{\text{brane}} + e^{b\phi/3}V_L\right] \ , \tag{2.8}$$

where $\tilde{g}_{mn} = G_{\mu\nu}\delta^\mu_m\delta^\nu_n$ is the induced metric on the domain wall, $\mathcal{L}_{\text{brane}}$ is the lagrangian of a gauge theory living an the brane and V_i the tensions of the branes. We thus get for the solution of the space with domain wall:

$$\begin{aligned} ds^2 &= [1 - N(a)|z|]^{32/a^2}\, d\vec{x}^2 - dz^2 \,, \\ e^{-2\phi} &= [1 - N(a)|z|]^{12/a} \,, \end{aligned} \tag{2.9}$$

The brane tensions V_i and the dilaton coupling b satisfy the matching conditions

$$V_0 = -V_L = \frac{8}{\kappa}\sqrt{\frac{3\Lambda}{a^2-64}}\,, \qquad b = \frac{1}{2}a\,. \tag{2.10}$$

V_0 (V_L) corresponds to the tension of the so-called Planck brane (TeV-brane). What we are actually doing by introducing these source terms is making an orbifold construction $S^1/\mathbb{Z}_2$, where the domain walls are located at the fix points. Note that we can also take the limit in which we send the TeV-brane to the singularity, by taking the limit $z_0 \to 0$.

In these variables the Randall-Sundrum (RS) limit $a \to 0$ is singular. To get a good picture of this limit, it is instructive to go to the conformal frame via the coordinate transformation

$$[1 - N(a)|z|]^{\frac{a^2-16}{a^2}} = [1 - O(a)|\omega|]\,, \tag{2.11}$$

where $|\omega|$ runs between 0 and $\omega_0 = O(a)^{-1}[1-(N(a)z_0)^{(a^2-16)/a^2}]$ and $O(a) = \frac{a^2-16}{a^2}N(a)$. In this frame the solution takes the form

$$
\begin{aligned}
ds^2 &= [1-O(a)|\omega|]^{\frac{32}{a^2-16}}\,(d\vec{x}^2 - d\omega^2)\,,\\
e^{-2\phi} &= [1-O(a)|\omega|]^{\frac{12a}{a^2-16}}\,.
\end{aligned} \tag{2.12}
$$

We can now make a case study for the different values of the dilaton couping a:

- In the conformal frame the RS limit $a \to 0$ is prefectly regular and gives us the non-dilatonic AdS_5 solution of RS [1, 2]. The graviton zero mode goes like $\psi(\omega) = [1-\sqrt{\Lambda/12}\,|\omega|]^{-3/2}$.

- For $0 < a^2 < 16$ we find the graviton zero mode falling off like

$$
\psi(\omega) = [1-O(a)|\omega|]^{\frac{24}{a^2-16}}\ , \tag{2.13}
$$

  i.e. confining faster and faster as a approaches the value -4. Note that the RS limit is the least confining case of this family.

- For $16 < a^2 < 64$ we find that the zero mode is normalizable only in a finite interval, even in the limit $\omega_0 \to O(a)^{-1}$.[1] However, we can always change coordinates and make the interval $[0,\omega_0]$ infinite. For example in coordinates $1-O(a)\hat{\omega} = (1-O(a)\omega)^{-1}$. Again the zero mode is confining

$$
\psi(\hat{\omega}) = [1-O(a)|\hat{\omega}|]^{\frac{-24}{a^2-16}}\ , \tag{2.14}
$$

  and the dilaton in these coordinates looks like

$$
e^{-2\phi} = [1-O(a)|\hat{\omega}|]^{\frac{-12a}{a^2-16}}\,. \tag{2.15}
$$

  Note that the exponent of the graviton zero mode is bigger (smaller) than in the RS case for values of $a < \sqrt{32}$ ($a > \sqrt{32}$). However, a comparison as in the previous case is difficult since after the coordinate transformation, we are no longer in the conformal frame.

Concluding, we find that in the any of the cases discussed above there is confinement of the gravitational zero mode, in some cases even stronger than in the RS-case. However there is a big difference in the behaviour of the dilaton: for positive values of a the exponent of the dilaton has the same sign as the exponent of the graviton, but for negative value of a the signs are opposite. Depending on the sign of a the string coupling constant e^{ϕ} at the space-time singularity goes to ∞ or zero.

Although the RS limir $a \to 0$ is well defined in the conformal frame (2.12), there is a singular point for the value $a = \pm 4$, which needs a special analysis. Solving the equations of motion for the $a = 4$ case, it becomes clear that there are only

[1] Note that by the definition of the range of ω, $\omega_0 = O(a)^{-1}$ is the maximal value it can attain.

two solutions: either $\Lambda = 0$ or linear dilaton with constant warp factor i.e. flat five-dimensional space-time metric. If we consider non-critical strings in five dimensions the cosmological constant term is given, in string frame, by $e^{-2\phi}(D_{cr} - D)/3$ with $D_{cr} = 26$ or 10. In this case the only solution is flat five-dimensional space-time and dilaton:

$$\phi = \frac{1}{2}\sqrt{\Lambda} z \,. \tag{2.16}$$

For $\Lambda = 0$ we find two solutions for two distict values of the coupling b, which only differ in the dilaton dependence:

$$b = -4 : \quad \begin{cases} ds^2 = \left[1 - \frac{2}{3}\kappa V_0 |z|\right]^{1/2} d\vec{x}^2 - dz^2 \,, \\ e^{-2\phi} = \left[1 - \frac{2}{3}\kappa V_0 |z|\right]^{-3/2} \,, \end{cases} \tag{2.17}$$

$$b = 4 : \quad \begin{cases} ds^2 = \left[1 - \frac{2}{3}\kappa V_0 |z|\right]^{1/2} d\vec{x}^2 - dz^2 \,, \\ e^{-2\phi} = \left[1 - \frac{2}{3}\kappa V_0 |z|\right]^{3/2} \,. \end{cases} \tag{2.18}$$

Here the coordinate z runs between 0 and $((\frac{2}{3}\kappa V_0)^{-1} - z_0)$ where $z_0 \in [0, (\frac{2}{3}\kappa V_0)^{-1}]$. In the conformal frame, these solutions are of the form:

$$b = -4 : \quad \begin{cases} ds^2 = \left[1 - \frac{1}{2}\kappa V_0 |\omega|\right]^{2/3} (d\vec{x}^2 - d\omega^2) \,, \\ e^{-2\phi} = \left[1 - \frac{1}{2}\kappa V_0 |\omega|\right]^{-2} \,, \end{cases} \tag{2.19}$$

$$b = 4 : \quad \begin{cases} ds^2 = \left[1 - \frac{1}{2}\kappa V_0 |\omega|\right]^{2/3} (d\vec{x}^2 - d\omega^2) \,, \\ e^{-2\phi} = \left[1 - \frac{1}{2}\kappa V_0 |\omega|\right]^{2} \,. \end{cases} \tag{2.20}$$

Again, as in the case of $16 < a^2 < 64$ above, we can always find a coordinate system in which the graviton zero mode is confined:

$$\psi(\hat{\omega}) = \left[1 - \frac{1}{2}\kappa V_0 |\hat{\omega}|\right]^{-1/2} \,. \tag{2.21}$$

The other singular point is at $a = -4$. This singularity however turns out to be a coordinate singularity due to the singular behaviour of the coordinate transformation (2.11). The solution at this point is given by

$$\begin{aligned} ds^2 &= e^{-\sqrt{-3/16\Lambda}|\omega|}\left(d\vec{x}^2 - d\omega\right) \,, \\ e^{-2\phi} &= e^{-\frac{3}{2}\sqrt{-3/16\Lambda}|\omega|} \,. \end{aligned} \tag{2.22}$$

Note that also in this case the graviton zero modes are confined.

3. Renormalisation group and Newton constant

Recently a different approach to RS-dynamics based on renormalization group interpretation of holography [14, 15] has been suggested in reference [12, 13]. In this approach the Einstein gravity on the wall is replaced by the integral on the ultraviolet region of the five-dimensional effective action. The four-dimensional cosmological constant remains fixed along the renormalization group evolution and therefore can be fine tuned to zero by impossing appropiated boundary conditions in the ultraviolet region. The solutions we have been describing above correspond to particular initial conditions determined by the values of the wall tension. Notice that this value fixed by the jump equations is independent of the particular value of the UV cutoff used to locate the wall i.e. it is renormalization group invariant.

In this renormalization group scheme we can define the following beta function:

$$\beta_\phi = A\frac{\partial}{\partial A}\phi\,, \tag{3.1}$$

or in terms of the "cosmological time":

$$\gamma = \frac{\dot{\phi}}{\beta_\phi}\,, \tag{3.2}$$

with

$$\gamma = \frac{\dot{A}}{A} \tag{3.3}$$

the expansion rate of the four-dimensional metric. For the solutions of dilatonic gravity we have:

$$\beta_\phi = -\frac{3}{8}a\,, \tag{3.4}$$

and we observe that the RS-model $a = 0$ corresponds to the conformal case $\beta_\phi = 0$ with all other cases constant but non-vanishing beta function (positive or negative depending on the sign of a). The naked singularity is characterized by infinite γ i.e. by $\dot{\phi} = \infty$. For the vanishing cosmological constant case we get the beta function:

$$\beta_\phi = \frac{3}{2}\,, \tag{3.5}$$

corresponding precisely to the point $a = 4$ i.e. to the singular line in figures 1 and 2.

Next we will study the evolution of the Newton constant. The relevant renormalization group equations is given by:

$$\left(\dot{A}\frac{\partial}{\partial A} + \dot{\phi}\frac{\partial}{\partial\phi}\right)\frac{1}{\kappa_4} = \frac{A^2}{\kappa_0}\,. \tag{3.6}$$

This equation have a very simple physical meaning. Namely the r.h.s. of the equation is simply the "time" derivative of the Newton constant κ_4 defined by Kaluza-Klein

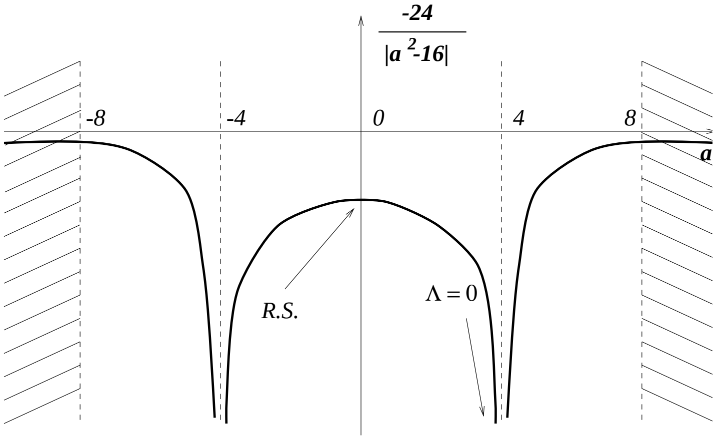

Figure 1: The graphic of the power of the graviton zero mode (2.13)–(2.14). There exists always a coordinate frame in which the zero mode can be described as being confined. In the region $-4 < a < 4$ this frame corresponds to the conformal one. The point $a = 0$ corrsponds to the Randall-Sundrum solution. The solution for $a = 4$ only exists for $\Lambda = 0$.

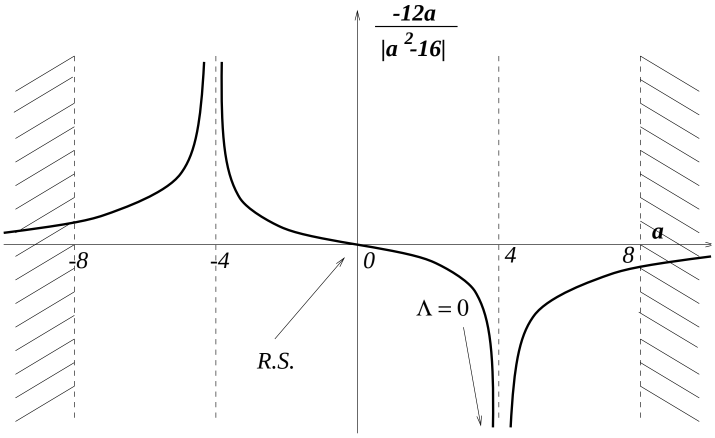

Figure 2: The graphic of the power of the dilaton (2.9)–(2.15). Again the region $-4 < a < 4$ corresponds to the conformal frame. For positive values of a the dilaton is proportional to the graviton, while for negative a's the dilaton is inversely proportional.

reduction on the bulk direction. Thus the meaning of the previous equation is simply that $\partial\kappa_4/\partial t = 0$.

This equation, once we have written $\dot{\phi}$ and $\dot{A}$ in terms of A has the the solution,

$$\kappa_4^{-1} = \frac{1}{\kappa}\int dz A^2(z) + \text{constant}\,. \tag{3.7}$$

For A as in (2.9), the above integral becomes

$$\kappa_4^{-1} = \frac{1}{\kappa}\int_0^{N(a)^{-1}-z_0} dz\,[1 - N(a)|z|]^{32/a^2}\;. \tag{3.8}$$

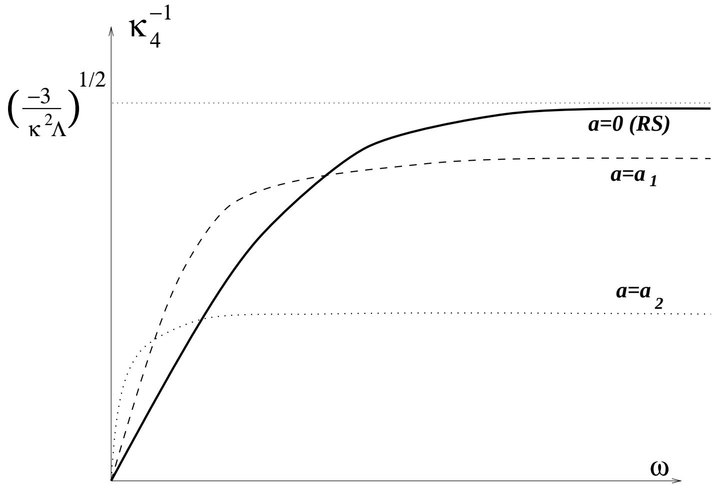

Figure 3: The running of the effective four-dimensional Newton constant as a function of the holographic distance ω for different values of the dilaton coupling $0 < a_1^2 < a_2^2 < 16$.

In order to be able to compare with the case of zero-dilaton (Randal-Sundrum), we switch to the conformal variable ω,

$$\kappa_4^{-1} = \frac{1}{\kappa}\int_0^{\omega_a} d\omega\, [1 - O(a)|\omega|]^{32/a^2} \;, \tag{3.9}$$

where the upper limit corresponds to the distance between the Planck brane and the place where the effective Newton constant is measured. For the case $a^2 < 16$, ω_0 runs in a semi-infinite range. Solving the integral (3.9) gives

$$\kappa_4^{-1} = \frac{1}{\sqrt{-3\kappa^2\Lambda}} \frac{12\sqrt{64 - a^2}}{32 + a^2}\left[1 - \Big(1 - O(a)\omega_0\Big)^{-\frac{32+a^2}{16-a^2}}\right], \tag{3.10}$$

which should be compared to the RS case:

$$\kappa_{4(RS)}^{-1} = \sqrt{\frac{-3}{\kappa^2\Lambda}}\left[1 - \left(1 + \sqrt{\frac{-\Lambda}{12}}\omega_0\right)^{-2}\right]. \tag{3.11}$$

On the other hand, it is not straightforward how to give an adequate comparison for case $16 < a^2 < 64$ and Randall-Sundrum. We have mentioned above that in the conformal frame the variable ω then runs over a finite range. This range can be made infinite, as done above, but again comparison to RS is hard due to the fact that we are no longer in the conformal frame.

The other physical implication is the screening of the measurements of physical quantities at the distance z_0 by the warp factor $A(z)$, generating a hierarchy between the Planck brane and the brane TeV-brane. In terms of the four and five-dimensional Planck-length the hierarchy is of the order of

$$\ell_4 \approx e^{-50}\ell_5 \,. \tag{3.12}$$

In the conformal frame (the only frame where we can compare to the RS case), we get

$$\omega_0 = -\frac{12}{a^2 - 16}\sqrt{\frac{a^2 - 64}{3\Lambda}}\left(e^{\frac{25}{8}(16-a^2)} - 1\right), \tag{3.13}$$

which should be compared to the Randal-Sundrum case,

$$\omega_0 = \sqrt{\frac{-12}{\Lambda}}\left(e^{50} - 1\right). \tag{3.14}$$

We see that the inclusion of the dilaton has a considerable effect on the effective Newton constant: the higher the values of dilaton coupling a^2, the faster the Newton constant reaches its asymptatic value. At the same time there is a screening of the constant which is bigger as the dilaton coupling grows.

Finally let us just mention a natural interpretation of the singularity from the four-dimensional physics point of view. This singularity, depending on the sign of β_ϕ could be interpreted either as a Landau pole or a confinement scale for the non-conformal gauge theory on the wall. It is interesting to see that this potential scale of the gauge theory is related with the four dimensional Newton scale.

Note added in proof: while this paper was being written we received the papers [16, 17]that partially overlap with our results.

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